Centripetal Acceleration

Introduction
By definition, an object that has a changing velocity is accelerating. So far we have dealt with situations in which acceleration meant changing speed, but not changing direction, or perhaps changing both speed and direction (i.e. projectile motion.) It is very possible to change your direction, but not your speed. Acceleration in this case is given the name *centripetal acceleration*.

Centripetal acceleration is the name given to an acceleration that only changes an object's direction of travel, but not its speed. This happens any time that an object's acceleration is perpendicular to its velocity. If an object has an acceleration of constant magnitude that is always perpendicular to its velocity, the object will travel in a circle with a constant speed. The equation

$$a_c = \frac{v^2}{r}$$

relates the magnitude of the acceleration (a_c) to the object's speed (v) and radius (r) of the circle.

Derivation of Equation
Diagram 1 below represents an object traveling counter-clockwise around a circle. It is traveling with a constant speed of V around a circle of radius R. An initial position and velocity are shown. Some time later, the "final" position and velocity are shown. The velocity vector will always be perpendicular to the position vector.

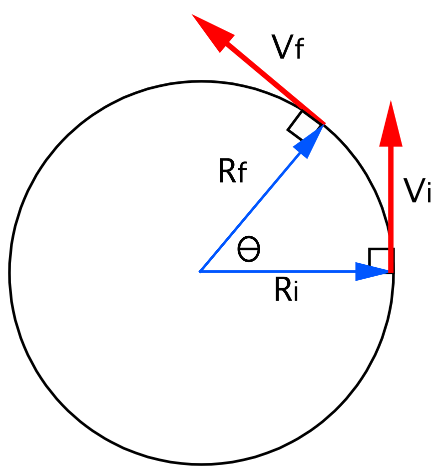

diagram 1

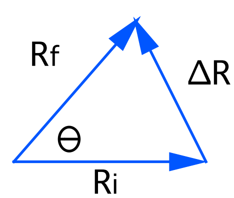

diagram 2

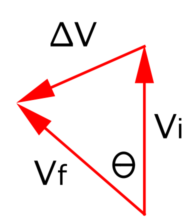

diagram 3

From this diagram, we can make two other diagrams, one for the position vectors (diagram 2), the other for the velocity vectors (diagram 3.) In both diagrams, the difference between the final and initial vectors is shown. These are similar triangles, so we can say

$$\frac{\Delta V}{V} = \frac{\Delta R}{R}$$

This is a scalar equation based on the magnitudes of the vectors drawn in the two vector diagrams. ΔV is the magnitude of the change in velocity, V is the speed, ΔR is the magnitude of the change in position, and R is the radius.

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We can say that the magnitude of the change in position is equal to the speed times time:

$$\Delta R = V\Delta t$$

This is true only if we take the limit as Δt approaches 0. ($V\Delta t$ is actually equal to the arc length that the object travels in Δt time. ΔR is the length of the chord that connects the initial and final positions. As Δt gets smaller, the chord length gets closer to the arc length; and are equal in the limit as Δt approaches zero.)

We can now say

$$\frac{\Delta V}{V} = \frac{V\Delta t}{R} \quad \text{as long as } \Delta t \to 0$$

Which we can re-write as

$$\frac{\Delta V}{\Delta t} = \frac{V^2}{R}$$

So we can say

$$\lim_{\Delta t \to 0} \frac{\Delta V}{\Delta t} = \frac{V^2}{R}$$

$$a_c = \frac{V^2}{R}$$

Where a_c is the centripetal acceleration. It is important to remember that this is a scalar, not a vector, equation. This equation tell us the magnitude of the acceleration, but not the direction.

To get the direction of the acceleration, we need to know the direction of the change in velocity. Looking back at our original position and velocity vector diagrams, we can see that we have isosceles triangles. Our equation was valid in the limit of Δt approaching zero. This means our diagrams show an isosceles triangle, with the angle θ approaching zero. Therefore, the other angles in the triangles approach 90^o; the instantaneous direction of ΔR is tangent to the circle and the instantaneous direction of ΔV is towards the center of the circle. So that means the velocity vector is tangent to the circle being traveled in and the acceleration vector is directed to the center of the circle.

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Derivation #2

This won't make sense until you know how to take the derivative of sines and cosines, but here is an algebreic way of doing the derivation.

In component notation, an object traveling in a circle with constant speed and constant radius would be written as

$$\vec{r} = R(\cos\omega t)\hat{i} + R(\sin\omega t)\hat{j}$$

In the above, R is the radius of the motion, and ω is the angular frequency of the motion. ω is related to the linear speed, v, by the relationship v = Rω. The position vector would always be pointing from the center of the circle to the object.

The velocity and acceleration are given by the first and second derivatives of the position:

$$\vec{v} = (-R\omega\sin\omega t)\hat{i} + (R\omega\cos\omega t)\hat{j}$$

$$\vec{a} = (-R\omega^2\cos\omega t)\hat{i} + (-R\omega^2\sin\omega t)\hat{j}$$

Notice that the acceleration can be rewritten as

$$\vec{a} = -\omega^2\vec{r}$$

The acceleration is always in the opposite direction of the position vector, so always points in to the center of the circle. The magnitude of the acceleration is then given by

$$a = R\omega^2$$

When we substitute the relationship ω=v/R, the acceleration can be written as

$$a = \frac{v^2}{R}$$